# HIDDEN UNITY IN THE QUANTUM DESCRIPTION OF MATTER

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We introduce an algebraic framework for interacting quantum systems that enables studying complex phenomena, characterized by the coexistence and competition of various broken symmetry states of matter. The approach unveils the hidden unity behind seemingly unrelated physical phenomena, thus establishing exact connections between them. This leads to the fundamental concept of *universality* of physical phenomena, a general concept not restricted to the domain of critical behavior. Key to our framework is the concept of *languages* and the construction of *dictionaries* relating them.

## 1 The Dictionaries of Nature

As Science progresses it tends to successfully describe the diverse phenomena encountered in nature with fewer underlying principles. Indeed, the search for the unifying principles behind the fundamental laws of physics is a common theme in the life of a physicist and has a very simple reason which is simplifying the understanding of the universe in which we live. Even if we knew the ultimate laws that govern the universe, could one predict the complex behavior observed in nature? This has been the subject of numerous works by very eminent people, like Philip W. Anderson, who rightfully argued that the whole is not necessarily the sum of its parts and thus “more is different.”[1] It seems as if matter organizes in well-defined but hard to decipher patterns.

The notion of symmetry has shaped our current conception of nature; however, nature is also full of symmetry breakings. Therefore understanding the idea of invariance and its corresponding conservation laws is as fundamental as determining the causes that prevent such harmony, and leads to more complex behavior. Unveiling and mastering the organizing principles is important since it leads, for example, to the design of new materials and devices with specific functionalities and unprecedented technological applications. Who would not like to have a room-temperature superconductor? However, the plethora of complex phenomena exceeds our ability to explain them, in part because of a lack of appropriate mathematical tools to disentangle its mysteries. Since quantum complexity is characterized by the coexistence and competition of various states of matter one needs an efficient and well-controlled approach to these problems that goes beyond traditional mean-field and semi-classical approximations.

Describing the structure and behavior of matter entails studying systems of interacting quantum constituents (bosons, fermions, spins) and essential to complexity are correlations, involving non-linear couplings, between their different components. In the quantum-mechanical description of matter, each physical system is naturally associated with a *language* of operators (for example, quantum spin-1/2 operators) and thus to an algebra realizing this language (e.g., the Pauli spin algebra generated by a family of commuting quantum spin-1/2 operators). It is

our point of view that crucial to the successful understanding of the mechanisms driving complexity is the realization of *dictionaries* (isomorphisms) connecting the different languages of nature and therefore linking seemingly unrelated physical phenomena. The existence of dictionaries provides not only a common ground to explore complexity but leads naturally to the fundamental concept of *universality*, meaning that different physical systems show the same behavior. In this way, there is a concept of physical equivalence hidden in these dictionaries.

In this chapter we present an algebraic framework for interacting extended quantum systems that allows studying complex phenomena characterized by the coexistence and competition of various broken symmetry states of matter. We show that exact algebraic and group theory methods are one of the most elegant and promising approaches towards a complete understanding of quantum phases of matter and their corresponding phase transitions. Previous to our work[4] there were two seemingly unrelated examples of dictionaries: The Jordan-Wigner (1928)[2] and Matsubara-Matsuda transformations (1956).[3] In addition to the generalization of these ($su(2)$) transformations to any irreducible spin representation, spatial dimension and particle statistics, we have proved a fundamental theorem which connects operators generating different algebras (e.g., $su(D)$ spin-particle connections), unifying the different languages known so far in the quantum-mechanical description of matter. The chapter has been written with the intention of providing the reader with the most fundamental concepts involved in our algebraic framework and how they apply to study complex phenomena. Much more details and examples can be found in the original manuscripts.[4–6]

## 2 Algebraic Approach to Interacting Quantum Systems

The theory of operator algebras on Hilbert spaces was initiated by Murray and von Neumann[7] as a tool to study unitary representations of groups, and as a framework for a reformulation of quantum mechanics. This area of research continued its development independently in the realm of mathematical physics, and therefore knowledge of those investigations remained bounded to specialists. For use of $C^*$ and $W^*$ algebras as a framework for quantum statistical mechanics one can look at the books of Bratteli and Robinson.[8] For the purposes of our presentation one only needs to have an elementary background in basic algebra,[9] and specially group theory.[10] In particular, Lie algebras and groups.

Here we are concerned with quantum lattice systems. A quantum lattice is identified with $\mathbb{Z}^{N_s}$, where $N_s$ is the total number of lattice sites (or modes). Associated to each lattice site $\mathbf{j} \in \mathbb{Z}^{N_s}$ there is a Hilbert space $\mathcal{H}_{\mathbf{j}}$ of finite dimension $D$ describing the "local" modes. The total Hilbert space is $\mathcal{H} = \bigotimes_{\mathbf{j}} \mathcal{H}_{\mathbf{j}}$. Since we are mostly interested in zero temperature properties, a state of the system is simply a vector $|\Psi\rangle$ in $\mathcal{H}$, and an observable is a selfadjoint operator $O : \mathcal{H} \rightarrow \mathcal{H}$. The dynamical evolution of the system is determined by its Hamiltonian $H$. The topology of the lattice, dictated by the connectivity and range of the interactions in $H$, is an important element in establishing complexity. In the case of quantum continuous systems we can still use the present formalism after discretizing the space. Going beyond this approach is outside the scope of these notes.

As mentioned above, each physical system is naturally associated with a language of operators, and thus to an algebra realizing this language. Formally, a *language* is defined by an operator algebra and a specific representation of the algebra. Mathematically, we use the following notation: *language* $= \mathcal{A} \wedge \Gamma_A$, where $\mathcal{A}$ is the operator algebra and $\Gamma_A$ is a particular irreducible representation (irrep) of the local algebra $\mathcal{A}_{\mathbf{j}}$ associated to $\mathcal{A}$, of dimension $\dim \Gamma_A = D$ (see Fig. 1).

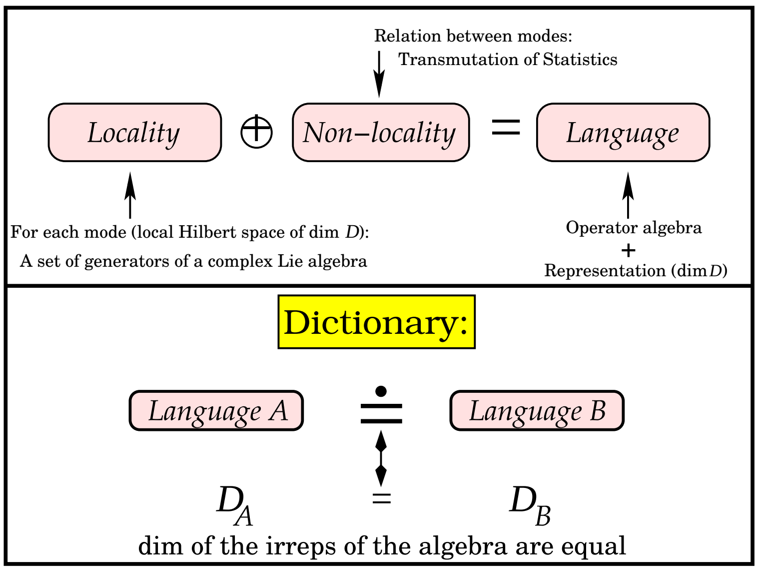


Figure 1. Definition of a *language* and theorem behind the construction of the dictionaries of nature. In the upper panel we show schematically what elements define a *language* $= \mathcal{A} \wedge \Gamma_A$, where $\mathcal{A}$ is the algebra and $\Gamma_A$ a particular irrep. In the lower panel we establish the criteria to build a dictionary, given two languages $A$ and $B$. This criteria is based upon Burnside's fundamental theorem of algebra.[9]

For the sake of clarity, let us choose the phenomenon of magnetism to illustrate the key ideas. This is one of the most intriguing and not fully understood problems in condensed matter physics where strong correlations between electrons (of electrostatic origin) are believed to be the essence of the problem. To describe the phenomenon using a minimal model (i.e., a model that only includes the relevant degrees of freedom) distinct approaches can be advocated depending upon the itineracy of the electrons that participate in the magnetic processes. In one extreme (e.g., insulators) a description in terms of localized quantum spins is appropriate, while in the other (e.g., metals) delocalization of the electrons is decisive and cannot be ignored. We immediately identify the languages associated to each description: quantum spins (e.g., Pauli algebra) and fermions (spin-1/2 Fermi algebra). Are these really different descriptions? Is there a dictionary that may connect the two languages? Let's assume that we decide to use the quantum spins language. What other seemingly unrelated phenomena are connected to magnetism? Can we re-

late phases of matter corresponding to dissimilar phenomena? Can an arbitrary physical system be mapped, for instance, onto a pure magnetic system (an array of quantum spins)?

In the following we will answer these questions by examples. A fundamental concept of universality, complementary to the one used in critical phenomena, emerges as a consequence of unveiling the hidden unity in the quantum-mechanical description of matter.

### *2.1 Building the Dictionaries of Nature*

The simplest model of magnetism is provided by the Heisenberg-Ising Hamiltonian

$$H = J \sum_{j=1}^{N_s-1} \Delta \; S^z_j S^z_{j+1} + \frac{1}{2} \; (S^+_j S^-_{j+1} + S^-_j S^+_{j+1}) \; , \tag{1}$$

where the operators $S^{\mu}_j$ satisfy an $\bigoplus_j su(2)$ algebra ($SU(2)$ symmetry of the Hamiltonian is recovered at the points $\Delta = \pm 1$). If we work in the $S = 1/2$ irrep it is well-known that the model can be mapped onto an interacting spinless fermion model through the Jordan-Wigner transformation.[2] More generally, the model can mapped onto[4,6]

$$H = J \sum_{j=1}^{N_s-1} \Delta (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}) + \frac{1}{2} \; (a^\dagger_j a_{j+1} + a^\dagger_{j+1} a_j) \tag{2}$$

through the transformation ($n_j = a^\dagger_j a_j$ and $0 \leq \theta < 2\pi$)

$$\begin{cases} S^+_j = a^\dagger_j \, K_j(\theta) \\ S^-_j = K^\dagger_j(\theta) \, a_j \\ S^z_j = n_j - \frac{1}{2} \end{cases} \; , \tag{3}$$

where the non-local statistical operator or transmutator $K_j(\theta)$ is given by

$$K_j(\theta) = e^{i\theta \sum_{i<j} n_i} = \prod_{i<j} [1 + (e^{i\theta} - 1) \; n_i] \tag{4}$$

since $n_j^2 = n_j$ (for any $\theta$), and satisfy $K_j(\theta) K^\dagger_j(\theta) = K^\dagger_j(\theta) K_j(\theta) = \mathbb{1}$. In this way we transformed the original localized spin-1/2 problem into an itinerant gas of (anyon) particles obeying the double-operator algebra ($[A, B]_\theta = AB - e^{i\theta} BA$)

$$\begin{cases} [a_i, a_j]_\theta = [a^\dagger_i, a^\dagger_j]_\theta = 0 \; , \\ [a_i, a^\dagger_j]_{-\theta} = \delta_{ij} (1 - (e^{-i\theta} + 1) n_j) \; , \; [n_i, a^\dagger_j] = \delta_{ij} a^\dagger_j \; , \end{cases} \tag{5}$$

(for $i \leq j$). Each statistical angle $\theta$ provides a different particle language and defines the exchange statistics of the particles. The case $\theta = \pi$ corresponds to canonical spinless fermions[2] while $\theta = 0$ represents hard-core (HC) bosons.[3] In all cases one can accommodate up to a single particle ($p = 1$) per quantum state, $(a^\dagger_j)^{p+1} = 0$ (i.e, the particles are "HC"). Figure 2 shows a classification of particles according to the independent concepts of exchange statistics and generalized Pauli exclusion principle.[6] The statistical operator connects particles within each equivalence class.

In order to construct a dictionary one also needs the inverse mapping

$$\begin{cases} a_j^\dagger = \prod_{i<j} [\frac{e^{-i\theta}+1}{2} + (e^{-i\theta}-1)\ S_i^z]\ S_j^+ \\ a_j = \prod_{i<j} [\frac{e^{i\theta}+1}{2} + (e^{i\theta}-1)\ S_i^z]\ S_j^- \\ n_j = S_j^z + \frac{1}{2} \end{cases} . \tag{6}$$

Let us consider now the same Hamiltonian, Eq. (1), but with spin operators in an $S=1$ irrep. A possible mapping in terms of two-flavor (or $s$=1/2) particles is[4]

$$\begin{cases} S_j^+ = \sqrt{2}\ (a_{j\uparrow}^\dagger\ K_j(\theta) + K_j^\dagger(\theta)\ a_{j\downarrow}) \\ S_j^- = \sqrt{2}\ (K_j^\dagger(\theta)\ a_{j\uparrow} + a_{j\downarrow}^\dagger\ K_j(\theta)) \\ S_j^z = n_{j\uparrow} - n_{j\downarrow} \end{cases} , \tag{7}$$

where the non-local transmutator ($n_{j\alpha} = a_{j\alpha}^\dagger a_{j\alpha}$, $n_j = n_{j\uparrow} + n_{j\downarrow}$, and $\alpha = \uparrow, \downarrow$)

$$K_j(\theta) = e^{i\theta \sum_{i<j} n_i} = \prod_{i<j} [1 + (e^{i\theta}-1)\ n_i] \tag{8}$$

($n_{j\alpha} n_{j\beta} = \delta_{\alpha\beta}\, n_{j\alpha}$) allows rotation of the statistics of the particles whose algebra is determined by ($i \leq j$)

$$\begin{cases} [a_{i\alpha}, a_{j\beta}]_\theta = [a_{i\alpha}^\dagger, a_{j\beta}^\dagger]_\theta = 0\ , \\ [a_{i\alpha}, a_{j\beta}^\dagger]_{-\theta} = \delta_{ij} \begin{cases} 1 - e^{-i\theta}\, n_{j\alpha} - n_j & \text{if } \alpha = \beta, \\ -e^{-i\theta}\, a_{j\beta}^\dagger a_{j\alpha} & \text{if } \alpha \neq \beta, \end{cases} \end{cases} . \tag{9}$$

In this case a more restrictive version of the Pauli exclusion principle applies where one can accommodate no more than a single particle per site regardless of $\alpha$, i.e., $a_{j\alpha}^\dagger a_{j\beta}^\dagger = 0,\ \forall(\alpha, \beta)$. In this language the resulting Hamiltonian is[11]

$$H = J \sum_{j=1}^{N_s-1} \Delta(n_{j\uparrow} - n_{j\downarrow})(n_{j+1\uparrow} - n_{j+1\downarrow}) + \sum_\alpha (a_{j\alpha}^\dagger a_{j+1\alpha} + a_{j\alpha}^\dagger a_{j+1\bar{\alpha}}^\dagger + \text{H.c.})\ . \tag{10}$$

One can define an inverse mapping as in the $S=1/2$ case, thus building the appropriate dictionary.[4] In a similar fashion, one could continue for higher spin $S$ irreps and would find that HC particles have $N_f = 2S$ flavors (we call these generalized Jordan-Wigner particles).[4] Of course, this is not the only way to proceed. For example, for half-odd integer cases where $2S+1 = 2^{\bar{N}_f}$ a simple transformation in terms of standard canonical multiflavor fermions is possible.[4,6]

What is the unifying concept behind the construction of the dictionaries of nature? When is it possible to build a dictionary between two arbitrary languages? The answers lie in the application of the following theorem together with the transmutation of statistics[6] (see lower panel in Fig. 1):
*Fundamental Theorem:* Given a set of generators of a complex Lie algebra $\tilde{\mathcal{L}}$ in a particular ($D$-dimensional) $\Gamma_A$ irrep, it is always possible to write these generators as a function of the identity and the generators of a bosonic language $\mathcal{B} \wedge \Gamma_B$ where

dim $\Gamma_B = D$. Similarly, each generator of the bosonic language can be written in terms of the identity and the generators of $\tilde{\mathcal{L}}$ in the $\Gamma_A$ irrep. The algebra defining a bosonic language is $\mathcal{B} = \bigoplus_{\mathbf{j}} \mathcal{B}_{\mathbf{j}}$, where the local algebra $\mathcal{B}_{\mathbf{j}}$ is a Lie algebra.

| Exclusion Principle ($p$) \ Statistics ($\theta$) | Bose | Anyon | Fermi |
|---|---|---|---|
| 1 | *hard–core bosons* | *hard–core anyons* | *canonical fermions* |
| 2 | | | |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $\infty$ | *canonical bosons* | | |

Figure 2. Classification of single-flavor "particles" according to the fundamental exclusion ($p$) and exchange statistics ($\theta$) principles. Here we only consider double-operator algebras; the general case of para-statistics (triple-operator algebras) is excluded. Each raw represents an equivalence class and we have used as representatives of those equivalence classes the known particles found in nature. Most fundamental is the concept of *language* which uniquely defines the type of particle.

This theorem provides the necessary and sufficient conditions to connect two bosonic languages. To construct dictionaries (isomorphisms) for arbitrary (bosonic, fermionic or anyonic) languages one needs to complement the theorem with the transmutators of statistics.[6] In our first example dim $\mathcal{H}_j = D = 2$ and the bosonic language is $\bigoplus_j su(2) \wedge S = \frac{1}{2}$, the fermionic and anyonic languages are obtained after application of the statistical transmutator. The second example only differs in the representation $(S = 1, D = 3)$.

In this section the emphasis was put in establishing the commonalities between the different languages. In addition, to illustrate the notion of coexistence and competition of phases one needs the solution of the model. In the next section we will explain the concept of universality by realizing a dictionary that connects different Lie algebras with $D = 3$ representations.

### *2.2 Unveiling Order behind Complexity*

The field of quantum phase transitions studies the changes that can occur in the macroscopic properties of matter at zero temperature due to changes in the parameters characterizing the system. To identify broken symmetry phases (ground states (GSs)) what is typically used as a working principle is Landau's postulate of an order parameter (OP). One of the major properties of this OP is the symmetry

rules it obeys. The space in which it resides is often represented as the quotient set of the symmetry group of the disordered phase and the symmetry group of the ordered phase. While one generally knows what to do if the OP is known, Landau's postulate gives no procedure for finding it. In this section we describe a simple algebraic framework for identifying OPs.

The system we want to study is the $SU(2)$-invariant model Hamiltonian ($J < 0$)

$$H_\vartheta = J\sqrt{2}\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\left[\cos\vartheta\ \mathbf{S_i}\cdot\mathbf{S_j} + \sin\vartheta\,(\mathbf{S_i}\cdot\mathbf{S_j})^2\right], \tag{11}$$

where $\mathbf{S_j}$ is a $S$=1 operator satisfying the algebra $\bigoplus_{\mathbf{j}} su(2)$, as before. Summation is over bonds $\langle \mathbf{i},\mathbf{j}\rangle$ of a regular $d$-dimensional lattice with $N_s$ sites and coordination $\mathsf{z}$. As we will see, for certain values of $\vartheta$ this Hamiltonian is highly symmetric: for $\vartheta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ it is explicitly invariant under uniform $SU(3)$ transformations on the spins, while for $\vartheta = \pm\frac{\pi}{2}$, it is explicitly invariant under staggered conjugate rotations of the two sublattices.[5]

The case $\vartheta = \frac{\pi}{4}$ can be conveniently written in the ($s = 1/2$) HC boson language of Eq. (7) with $\theta = 0$

$$H_{\frac{\pi}{4}} = J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}\sum_{\alpha}\left(a^\dagger_{\mathbf{i}\alpha}a_{\mathbf{j}\alpha} + \text{H.c.}\right) + 2\mathbf{s_i}\cdot\mathbf{s_j} + 2\left(1 - \frac{n_\mathbf{i}+n_\mathbf{j}}{2} + \frac{3}{4}n_\mathbf{i}n_\mathbf{j}\right)\ , \tag{12}$$

where $\mathbf{s_j} = \frac{1}{2}a^\dagger_{\mathbf{j}\mu}\boldsymbol{\sigma}_{\mu\nu}a_{\mathbf{j}\nu}$ with $\boldsymbol{\sigma}$ denoting Pauli matrices.

For a system of $\mathcal{N} = \mathcal{N}_\uparrow + \mathcal{N}_\downarrow$ ($\mathcal{N} \leq N_s$) HC bosons the exact GS is[5]

$$|\Psi_0(\mathcal{N}, S_z)\rangle = (\tilde{a}^\dagger_{\mathbf{0}\uparrow})^{\mathcal{N}_\uparrow}(\tilde{a}^\dagger_{\mathbf{0}\downarrow})^{\mathcal{N}_\downarrow}|0\rangle\ , \tag{13}$$

with an energy $E_0/N_s = J\mathsf{z}$ and a total $S_z = \frac{\mathcal{N}_\uparrow - \mathcal{N}_\downarrow}{2}$. The operator $\tilde{a}^\dagger_{\mathbf{0}\alpha}$ is the $\mathbf{k} = \mathbf{0}$ component of $a^\dagger_{\mathbf{j}\alpha}$, i.e., $\tilde{a}^\dagger_{\mathbf{k}\alpha} = \frac{1}{\sqrt{N_s}}\sum_{\mathbf{j}}\exp[i\mathbf{k}\cdot\mathbf{r_j}]\ a^\dagger_{\mathbf{j}\alpha}$. The quasihole and quasiparticle excited states are

$$\begin{cases} |\Psi^h_\mathbf{k}(\mathcal{N}, S_z)\rangle = \tilde{a}_{\mathbf{k}\alpha}|\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasihole,} \\ |\Psi^p_\mathbf{k}(\mathcal{N}, S_z)\rangle = \tilde{a}^\dagger_{\mathbf{k}\alpha}|\Psi_0(\mathcal{N}, S_z)\rangle & \text{quasiparticle,} \end{cases} \tag{14}$$

with the excitation energy of each being $\omega_\mathbf{k} = J\mathsf{z}(\frac{1}{\mathsf{z}}\sum_\nu e^{i\mathbf{k}\cdot\mathbf{e}_\nu} - 1)$ where the sum runs over the vectors $\mathbf{e}_\nu$ which connect a given site to its $\mathsf{z}$ nearest neighbors. In the $|\mathbf{k}| \to 0$ limit, $\omega_\mathbf{k} \to 0$.

Clearly the GS in Eq. (13) is a ferromagnetic Bose-Einstein (BE) condensate with arbitrary spin polarization, and the form of the result is *independent of the spatial dimensionality of the lattice*. We note that different values of $S_z$ correspond to the different orientations of the magnetization $\mathcal{M}$ associated to the broken $SU(2)$ spin rotational symmetry of the GS. We also note that the degeneracy of states with different number of particles $\mathcal{N}$ indicates a broken $U(1)$ charge symmetry (conservation of the number of particles) associated to the BE condensate. A signature of Bose condensation is the existence of off-diagonal long-range order (ODLRO) in the correlation function $\Phi_{\alpha\beta}(\mathbf{ij}) = \langle a^\dagger_{\mathbf{i}\alpha}a_{\mathbf{j}\beta}\rangle$ since that implies that there is at least one eigenvector with an eigenvalue of order $N_s$.[12]

We can easily compute the magnetization $\mathcal{M}$ and phase coherence of these various (non-normalized) degenerate GSs for a given density $\rho = \frac{\mathcal{N}}{N_s}$. For example, in the fully polarized case, $\mathcal{N} = \mathcal{N}_\uparrow$, $\mathcal{M} = \langle S^z_{\mathbf{j}} \rangle = \rho$, and the ODLRO ($\mathbf{r_i} \neq \mathbf{r_j}$) $\Phi_{\uparrow\uparrow}(\mathbf{ij}) = \frac{\rho(1-\rho)}{1-\epsilon}$, where $\epsilon = 1/N_s$. Similarly, the two-particle correlation function $\langle \Delta^\dagger_{\mathbf{i}} \Delta_{\mathbf{j}} \rangle = \Phi_{\uparrow\uparrow}(\mathbf{ij}) \frac{(\rho-\epsilon)(1-\rho-\epsilon)}{(1-2\epsilon)(1-3\epsilon)}$, where $\Delta^\dagger_{\mathbf{i}} = a^\dagger_{\mathbf{i}\uparrow} a^\dagger_{\mathbf{i}+\boldsymbol{\delta}\uparrow}$.[5]

The exact solution defines the features of the phase diagram that our proposed framework must qualitatively admit. We will see now that both OPs (magnetization and phase) are embedded in an $SU(3)$ OP. To this end one introduces a new language based upon the $\bigoplus_{\mathbf{j}} su(3)$ algebra in the fundamental representation with generators satisfying ($\mu, \nu \in [0,2]$) $[\mathcal{S}^{\mu\mu'}(\mathbf{j}), \mathcal{S}^{\nu\nu'}(\mathbf{j})] = \delta_{\mu'\nu} \mathcal{S}^{\mu\nu'}(\mathbf{j}) - \delta_{\mu\nu'} \mathcal{S}^{\nu\mu'}(\mathbf{j})$. One can rewrite Eq. (11) (up to an irrelevant constant) in this new language as

$$H_\vartheta = J\sqrt{2} \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \left[ \cos\vartheta \ \mathcal{S}^{\mu\nu}(\mathbf{i}) \mathcal{S}^{\nu\mu}(\mathbf{j}) + (\sin\vartheta - \cos\vartheta) \ \mathcal{S}^{\mu\nu}(\mathbf{i}) \tilde{\mathcal{S}}^{\nu\mu}(\mathbf{j}) \right] , \qquad (15)$$

where $\mathcal{S}(\mathbf{j})$ defines the $su(3)$ spin-particle mapping

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - n_{\mathbf{j}} & a_{\mathbf{j}\uparrow} & a_{\mathbf{j}\downarrow} \\ a^\dagger_{\mathbf{j}\uparrow} & n_{\mathbf{j}\uparrow} - \frac{1}{3} & a^\dagger_{\mathbf{j}\uparrow} a_{\mathbf{j}\downarrow} \\ a^\dagger_{\mathbf{j}\downarrow} & a^\dagger_{\mathbf{j}\downarrow} a_{\mathbf{j}\uparrow} & n_{\mathbf{j}\downarrow} - \frac{1}{3} \end{pmatrix} , \quad \tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - n_{\mathbf{j}} & -a^\dagger_{\mathbf{j}\downarrow} & -a^\dagger_{\mathbf{j}\uparrow} \\ -a_{\mathbf{j}\downarrow} & n_{\mathbf{j}\downarrow} - \frac{1}{3} & a^\dagger_{\mathbf{j}\uparrow} a_{\mathbf{j}\downarrow} \\ -a_{\mathbf{j}\uparrow} & a^\dagger_{\mathbf{j}\downarrow} a_{\mathbf{j}\uparrow} & n_{\mathbf{j}\uparrow} - \frac{1}{3} \end{pmatrix} . \qquad (16)$$

$\tilde{\mathcal{S}}(\mathbf{j})$ generates the conjugate representation. For $\vartheta = \frac{\pi}{4}$ we easily recognize the $SU(3)$ symmetric Heisenberg model and when $J < 0$ the GS is the state with maximum total $SU(3)$ spin $\mathcal{S}$. The OP associated with this broken symmetry is the total $SU(3)$ magnetization

$$\mathcal{S}^{\mu\nu}(\mathbf{k}) = \frac{1}{N_s} \sum_{\mathbf{j}} e^{i\mathbf{k}\cdot\mathbf{r_j}} \ \mathcal{S}^{\mu\nu}(\mathbf{j}) \qquad (17)$$

which has eight independent components. When $\langle \mathcal{S}^{\mu\nu} \rangle \neq 0$, the system orders, and the coexistence of a ferromagnetic phase and a BE condensation becomes more evident: In the HC boson language both OPs correspond to different components of the $SU(3)$ OP (see Eq. (16)). Table 1 summarizes the relations between OPs and quantum phases in the different languages for the homogeneous $\mathbf{k} = \mathbf{0}$ case.

A concept of universality naturally emerges from the dictionaries: Many apparently different problems in nature have the same underlying algebraic structure and, therefore, the same physical behavior. If it is the whole system Hamiltonian that maps onto another in a different language (like the example we described above), the universality applies to all length and time scales. However, sometimes only particular invariant subspaces of the original Hamiltonian map onto another system Hamiltonian. In this case, universality is only manifested at certain energy scales. The $t$-$J_z$ chain model provides a beautiful example of the latter situation:[13] the low-energy manifold of states maps onto an $XXZ$ model Hamiltonian, which can be exactly solved using the Bethe ansatz.

The fact that two dissimilar physical phenomena share the same set of critical exponents is also known as universality. Those critical phenomena are grouped into universality classes. Members of a given universality class have the same broken

Table 1. Generators of OPs and its relations for three different languages $\mathcal{A} \wedge \Gamma_A$. Each column represents a language, in this case dim $\Gamma_A = D = 3$. M stands for magnetism, SN spin-nematic, BE Bose-Einstein condensation, and CDW charge-density wave. $su(3)\wedge$ FR is the hierarchical language with FR meaning fundamental representation.

| $su(2) \wedge S = 1$ | HC bosons $\wedge\ \alpha = 2$ | $su(3) \wedge$ FR |
|---|---|---|
| M $\begin{cases} S^x = \frac{1}{\sqrt{2}}(\mathcal{S}^{01} + \mathcal{S}^{20} + \mathcal{S}^{02} + \mathcal{S}^{10}) \\ S^y = \frac{-1}{\sqrt{2}i}(\mathcal{S}^{01} + \mathcal{S}^{20} - \mathcal{S}^{02} - \mathcal{S}^{10}) \\ S^z = \mathcal{S}^{11} - \mathcal{S}^{22} \end{cases}$<br>SN $\begin{cases} (S^x)^2 = \frac{2}{3} + \frac{1}{2}(\mathcal{S}^{12} + \mathcal{S}^{21} + \mathcal{S}^{00}) \\ (S^z)^2 = \frac{2}{3} - \mathcal{S}^{00} \\ \{S^x, S^y\} = i(\mathcal{S}^{21} - \mathcal{S}^{12}) \\ \{S^x, S^z\} = \frac{1}{\sqrt{2}}(\mathcal{S}^{01} - \mathcal{S}^{20} - \mathcal{S}^{02} + \mathcal{S}^{10}) \\ \{S^y, S^z\} = \frac{-1}{\sqrt{2}i}(\mathcal{S}^{01} - \mathcal{S}^{20} + \mathcal{S}^{02} - \mathcal{S}^{10}) \end{cases}$ | M $\begin{cases} s^x = \frac{1}{2}(\mathcal{S}^{12} + \mathcal{S}^{21}) \\ s^y = \frac{1}{2i}(\mathcal{S}^{12} - \mathcal{S}^{21}) \\ s^z = \frac{1}{2}(\mathcal{S}^{11} - \mathcal{S}^{22}) \end{cases}$<br>BE $\begin{cases} a^\dagger_\uparrow = \mathcal{S}^{10} \\ a^\dagger_\downarrow = \mathcal{S}^{20} \\ a_\uparrow = \mathcal{S}^{01} \\ a_\downarrow = \mathcal{S}^{02} \end{cases}$<br>CDW $\begin{cases} n = \frac{2}{3} - \mathcal{S}^{00} \end{cases}$ | $\mathcal{S}^{\mu\nu}$<br>$\mu, \nu \in [0, 2]$<br>$\mathrm{Tr}\, \mathcal{S} = 0$ |

symmetry group (OPs), and the long-wavelenght excitations are described by a unique fixed-point Hamiltonian. The idea of universality that emerges from our work is complementary to the one used to analyze critical behavior. It is not restricted to the study of critical phenomena, but can be exploited in conjunction with Renormalization Group techniques.

## 3 Concluding Remarks

We presented an algebraic framework aimed at uncovering the order behind the potential multiplicity of complex phases in interacting quantum systems, a new paradigm at the frontiers of condensed matter physics. Crucial to this approach is the existence of dictionaries (isomorphisms) that permits to connect the different languages used in the quantum-mechanical description of matter. We also introduced the idea of universality of physical phenomena, a concept that naturally emerges from those dictionaries. In all cases we have given precise mathematical definitions to these physical terms.

The development of exact algebraic methods is one of the most elegant and promising tools towards the complete understanding of quantum phases of matter and their corresponding phase transitions. Often these systems are near quantum criticality which makes their study extremely complicated, if not impossible, for the traditional techniques such as mean-field or perturbative schemes. Precisely the same reason which prevents the use of these theories is the key for the successful application of algebraic methods: The absence of a small parameter and degeneracy for different quantum complex orderings.

There are several reasons why our algebraic framework constitutes a powerful

method to study complex phenomena in interacting quantum systems. Most importantly: To connect seemingly unrelated physical phenomena (e.g., models for high-$T_c$ or heavy fermion systems and quantum spin theories); identify the general symmetry principles behind complex phase diagrams; unveil hidden symmetries (and associated order parameters) to explore new states of matter with internal orders not envisaged before; obtain exact solutions of relevant physical models that display complex ordering at certain points in Hamiltonian space.

The algebraic framework for identifying OPs and possible broken symmetry phases of quantum systems can be summarized as follows:[6]

• Identify the dimension $D$ of the local Hilbert space $\mathcal{H}_{\mathbf{j}}$ which sets the dimension of the irrep $\Gamma_A$ associated to the language A.

• The OP is constructed from a hierarchical group. A hierarchical language is one whose local algebra $\mathcal{A}_{\mathbf{j}}$ has as fundamental representation of dimension $D$.

• Identify the embedded subgroups. Reduce the fundamental representation of the hierarchical group according to the irreps of the embedded subgroups, thereby establishing a hierarchical classification of the OPs and an enumeration of the possible broken symmetry phases.

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11. At this point two comments are in order. The first refers to the irrelevance of particle statistics for one-dimensional models with nearest-neighbor interactions like Eq. (1). Clearly one needs two or higher spatial dimensions to observe its effect, or non-local interactions; indeed, the relevance of exchange statistics is related to the connectivity of the lattice. The second comment is the observation that similar to the non-local rotation of statistics $\theta$ one could define local rotations between different flavors.[6]
12. In general (since there is a continuous $SU(2)$ symmetry, apart from the $U(1)$, that is broken) when $\mathcal{N}_\uparrow$ and $\mathcal{N}_\downarrow$ are both order $N_s$, there will be two eigenvectors with eigenvalues of order $N_s$ and, therefore, the condensate is a mixture.

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